# Mathematics
# Higher level
# Paper 3 – sets, relations and groups

Thursday 21 May 2015 (afternoon)

1 hour

## Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[60 marks]**.

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Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

**1.** [Maximum mark: 13]

Consider the set $S_3 = \{p, q, r, s, t, u\}$ of permutations of the elements of the set $\{1, 2, 3\}$, defined by

$$p = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},\ q = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},\ r = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$

$$s = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},\ t = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},\ u = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Let $\circ$ denote composition of permutations, so $a \circ b$ means $b$ followed by $a$. You may assume that $(S_3, \circ)$ forms a group.

(a) Complete the following Cayley table

| $\circ$ | $\boldsymbol{p}$ | $\boldsymbol{q}$ | $\boldsymbol{r}$ | $\boldsymbol{s}$ | $\boldsymbol{t}$ | $\boldsymbol{u}$ |
|---|---|---|---|---|---|---|
| $\boldsymbol{p}$ | | | | | | |
| $\boldsymbol{q}$ | | | $t$ | | | $s$ |
| $\boldsymbol{r}$ | | $u$ | | $t$ | $s$ | $q$ |
| $\boldsymbol{s}$ | | $t$ | $u$ | | | $r$ |
| $\boldsymbol{t}$ | | $s$ | $q$ | $r$ | | |
| $\boldsymbol{u}$ | | $r$ | $s$ | $q$ | | |

[5]

(b) (i) State the inverse of each element.

(ii) Determine the order of each element. [6]

(c) Write down the subgroups containing

(i) $r$,

(ii) $u$. [2]

**2.** [Maximum mark: 10]

The binary operation $*$ is defined for $x, y \in S = \{0, 1, 2, 3, 4, 5, 6\}$ by

$$x * y = \left(x^3 y - xy\right) \bmod 7 .$$

(a) Find the element $e$ such that $e * y = y$, for all $y \in S$. [2]

(b) (i) Find the least solution of $x * x = e$.

(ii) Deduce that $(S, *)$ is not a group. [5]

(c) Determine whether or not $e$ is an identity element. [3]

**3.** [Maximum mark: 12]

The relation $R$ is defined on $\mathbb{Z}$ by $xRy$ if and only if $x^2 y \equiv y \bmod 6$.

(a) Show that the product of three consecutive integers is divisible by $6$. [2]

(b) Hence prove that $R$ is reflexive. [3]

(c) Find the set of all $y$ for which $5Ry$. [3]

(d) Find the set of all $y$ for which $3Ry$. [2]

(e) Using your answers for (c) and (d) show that $R$ is not symmetric. [2]

**4.** [Maximum mark: 9]

Let $X$ and $Y$ be sets. The functions $f : X \to Y$ and $g : Y \to X$ are such that $g \circ f$ is the identity function on $X$.

(a) Prove that

(i) $f$ is an injection,

(ii) $g$ is a surjection. [6]

(b) Given that $X = \mathbb{R}^+ \cup \{0\}$ and $Y = \mathbb{R}$, choose a suitable pair of functions $f$ and $g$ to show that $g$ is not necessarily a bijection. [3]

**Turn over**

**5.** [Maximum mark: 16]

Consider the sets

$$G = \left\{ \frac{n}{6^i} \mid n \in \mathbb{Z},\ i \in \mathbb{N} \right\},\ H = \left\{ \frac{m}{3^j} \mid m \in \mathbb{Z},\ j \in \mathbb{N} \right\}.$$

(a) Show that $(G, +)$ forms a group where $+$ denotes addition on $\mathbb{Q}$. Associativity may be assumed. [5]

(b) Assuming that $(H, +)$ forms a group, show that it is a proper subgroup of $(G, +)$. [4]

The mapping $\phi : G \to G$ is given by $\phi(g) = g + g$, for $g \in G$.

(c) Prove that $\phi$ is an isomorphism. [7]